# Synthetic Algebraic Geometry

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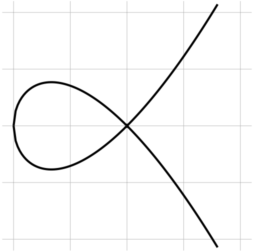
Interactions of Proof Assistants and Mathematics
Regensburg, 2023-09-19

$$X^3 + X^2 - Y^2 = 0$$



# The spectrum of an algebra

Let $R$ be a ring.

## Definition

If $A$ is a finitely presented $R$-algebra, we define:

$$\mathrm{Spec}(A) := \mathrm{Hom}_{R\text{-Alg}}(A, R)$$

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Spec : CommAlgebra R ℓ' → Type _
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- $\mathrm{Spec}(R[X]) = R$
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$$A \to R^{\mathrm{Spec}(A)}$$

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**This is counter-classical**

$$(\forall x{:}R.\, x = 0 \vee x \neq 0) \implies R = 0$$

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## Axiom (Loc)

$R$ is a local ring. That is: $0 \neq_R 1$ and $\forall x{:}R.\,(x \text{ invertible}) \vee (1 - x \text{ invertible})$.

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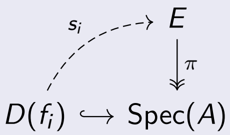
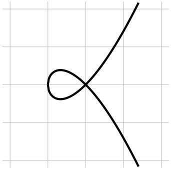
## Axiom (Z-choice)

Every surjection $\pi$ *locally* has sections:



# A secret connection between two equations



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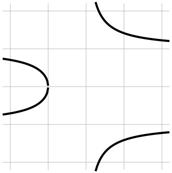

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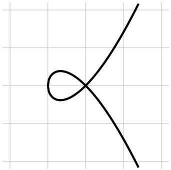

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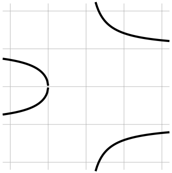

$Z = 1$ $\qquad$ $X = 1$

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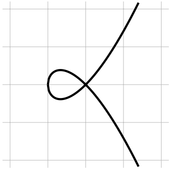

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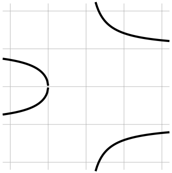

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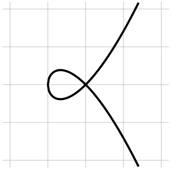

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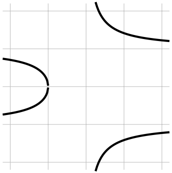

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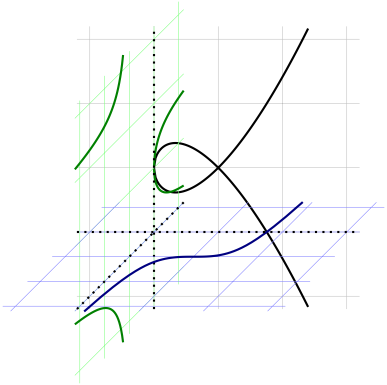
This is a homogenous equation.
Its solutions are best described as ratios $[x : y : z]$.

# Projective space as a scheme



## Definition

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where $x \sim y :\Leftrightarrow \exists a{:}R^\times.\ ax = y$.

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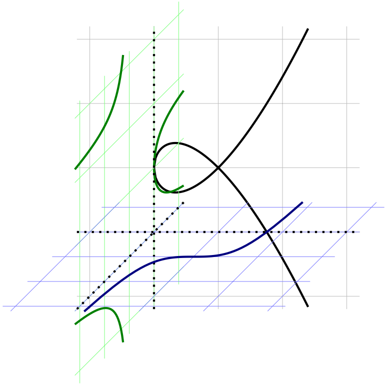

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**Definition**

A set $X$ is a *scheme* if it can be covered by finitely many subsets $U_i \subseteq X$ such that:

- Every $U_i$ is of the form $\mathrm{Spec}(A)$.
- For all $i$ and $x : X$, there exist $a_1, \ldots, a_n : R$ such that

$$x \in U_i \Leftrightarrow (a_1 \text{ inv.}) \vee \cdots \vee (a_n \text{ inv.})$$

**Example**

Define $U_i \subseteq \mathbb{P}^n$ by

$$[x] \in U_i :\Leftrightarrow (x_i \text{ invertible}).$$

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## Vision

synthetic language = external language + axioms

# Conclusion

- Synthetic languages are powerful abstraction layers.
- Please formalize constructively.
- Algebraic geometry is so much *fun* when you do it synthetically!

**Preprint:**

https://arxiv.org/abs/2307.00073

**(Very partial) formalization:**

https://github.com/felixwellen/synthetic-geometry